CHIBA-EP-95
OUTP-96-31P
hep-th/9608133
June 1996

An Exact Solution of Gauged Thirring Model in Two Dimensions *

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Abstract

In two space-time dimensions, we write down the exact and closed Schwinger-Dyson equation for the gauged Thirring model which has been proposed recently by the author. The gauged Thirring model is a natural gauge-invariant extension of the Thirring model and reduces to the Schwinger model (in the Abelian case) in the strong four-fermion coupling limit. The exact SD equation is derived by making use of the transverse Ward-Takahashi identity as well as the usual (longitudinal) Ward-Takahashi identity. Moreover the exact solution of the SD equation for the fermion propagator is obtained together with the vertex function in the Abelian gauged case. Finally we discuss the dynamical fermion mass generation based on the solution of the SD equation.

Key words: Thirring model, Schwinger model, exact solution, Schwinger-Dyson equation, Ward-Takahashi identity, dynamical mass generation

PACS: 11.10.Kk, 11.15.-q, 11.15.Tk, 12.38.Lg

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1 Introduction

It is well known that the existence of some symmetry in quantum field theory leads to the various identities which hold among the Green functions and the vertex functions, which are generally called the Ward-Takahashi (WT) identities [1]. In the previous paper [2], we have rederived, based on the path integral formalism, a new type of WT identities, so-called the transverse WT identities [3], for the vector current and the axial vector current. We have suggested to use the transverse WT identity as well as the usual (longitudinal) WT identity to specify the vertex function in the Schwinger-Dyson (SD) equation in the gauge theory. Especially, we have shown that they lead to the exact and closed SD equation for the fermion propagator in Abelian gauge theory defined on the 1+1 dimensional Minkowski space-time, when the bare fermion mass m_0 is zero (chiral limit). Actually, the formal solution of the SD equation obtained in such a way in massless QED_2 agrees with the exact solution obtained from the path integral formalism. Therefore, the framework of the SD equation supplemented with the transverse as well as the longitudinal WT identities can give the exact solution at least for the Abelian gauge theory in 1+1 dimensional Minkowski space-time.

In this paper we apply this strategy to solve the Thirring model [4] and the gauged Thirring model [5] in 1+1 dimensions. First of all, we must rewrite the Thirring model as a gauge theory. Such a procedure has been recently presented, first from the viewpoint of the hidden local symmetry [6] and then from the constrained system of Batalin-Fradkin type, see [7, 8, 9]. In this reformulation, the original Thirring model can be regarded as a gauge-fixed version (unitary gauge) of the reformulated Thirring model as a gauge theory. The auxiliary vector field (which is originally introduced into the theory in order to bi-linearize the four-fermion interaction) can be identified with the gauge-boson field in this reformulation, although the corresponding kinetic term is absent in this stage. On the other hand, it has been shown that the kinetic term for the gauge boson field can be generated due to radiative corrections [10, 6]. Quite recently, the Thirring model as a gauge theory has been further extended so as to include the kinetic term for the gauge field from the very beginning, which we call the gauged Thirring model [5]. In the gauged Thirring model, the Thirring model is recovered in the strong gauge-coupling limit $e^2 \to \infty$. On the other hand, the strong four-fermion coupling limit $G_T \to \infty$ reduces to the QED or QCD when the gauge theory in question is Abelian type or non-Abelian type, respectively.

Then, applying the reformulation of the Thirring model as a gauge theory to the 1+1 dimensional case, we can write down the exact SD equation for the fermion propagator, according to the procedure proposed in the previous paper [2]. In this framework the exact formal solution is obtained (only) in the chiral limit $m_0 = 0$. This formal solution has no extra pole at $p^2 \neq 0$ which implies that the fermion remains massless for $m_0 = 0$, which is the same situation as the massless Schwinger model (QED_2) [11]. As we mentioned in the above, QED_2 is regarded as the strong four-fermion coupling limit $G_T \to \infty$ of the Abelian-gauged Thirring model, in which the dynamical fermion mass generation is expected to occur. Therefore, we expect that the SD equation should have a solution corresponding to the dynamical fermion mass generation even in the limit $m_0 = 0$. From this viewpoint, we study the dynamical

fermion mass generation in the pure Thirring model limit $e^2 \to \infty$ and give perspectives of future studies for the gauged Thirring model with finite e^2 ($0 < e^2 < \infty$).

2 Thirring model as a gauge theory

We consider the Thirring model with the Lagrangian

$$\mathcal{L}_{Th} = \bar{\Psi}^j i\gamma^\mu \partial_\mu \Psi^j - \bar{\Psi}^j \hat{m}_0 \Psi^j - \frac{G_T}{2}(\bar{\Psi}^j \gamma_\mu \Psi^j)(\bar{\Psi}^k \gamma^\mu \Psi^k), \tag{2.1}$$

where Ψ^j is a Dirac fermion with a flavor index j running from 1 to N_f and $\hat{m}_0$ denotes the mass matrix for the fermion. By introducing the auxiliary vector field A_μ, the Thirring model is rewritten as

$$\mathcal{L}_{Th'} = \bar{\Psi}^j i\gamma^\mu (\partial_\mu - iA_\mu)\Psi^j - \bar{\Psi}^j \hat{m}_0 \Psi^j + \frac{1}{2G_T} A_\mu^2. \tag{2.2}$$

By introducing a scalar field θ, we can write down a gauge-invariant generalization of the Thirring model as [5]

$$\mathcal{L}_{Th''} = \bar{\Psi}^j i\gamma^\mu (\partial_\mu - iA_\mu)\Psi^j - \bar{\Psi}^j \hat{m}_0 \Psi^j + \frac{1}{2G_T}(A_\mu - \partial_\mu \theta)^2 - \frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu}, \tag{2.3}$$

where we have also included the kinetic term for the vector field A_μ with a gauge coupling constant e. Indeed, the Lagrangian (2.3) is invariant under the local $U(1)$ gauge transformation:

$$\begin{aligned} \Psi(x) &\to \Psi(x) e^{i\omega(x)}, \\ A_\mu(x) &\to A_\mu(x) + \partial_\mu \omega(x), \\ \theta(x) &\to \theta(x) + \omega(x), \quad (\phi(x) \to \phi(x) e^{i\omega(x)}) \end{aligned} \tag{2.4}$$

for a new variable $\phi(x) = e^{i\theta(x)}$. [1]

The original Thirring model is identified with the strong gauge-coupling limit $e^2 = \infty$ of a gauge-fixed version of the gauge theory with the Lagrangian (2.3), which we call the *(Abelian-)gauged Thirring model* [5]. Indeed, the Lagrangian (2.3) reduces to the Lagrangian (2.2), if we take the unitary gauge: $\theta(x) \equiv 0$ and set $e^2 = \infty$. However, actual calculations such as loop calculations are generally impossible in the unitary gauge. For such purposes the covariant gauge is most convenient, although both theories should give the same results on the gauge-invariant quantities, e.g., the chiral condensate $\langle \bar{\Psi}\Psi \rangle$.

[1] The gauged Thirring model can also be regarded with the Higgs-Kibble model (or gauged non-linear sigma model) in the presence of fermions:

$$\mathcal{L}_{Th'''} = \bar{\Psi}^j i\gamma^\mu (\partial_\mu - iA_\mu)\Psi^j - \bar{\Psi}^j \hat{m}_0 \Psi^j + \frac{1}{2G_T}|(\partial_\mu + iA_\mu)\phi|^2 - \frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu}. \tag{2.5}$$

Now we briefly review the covariantly gauge-fixed BRST formulation of the gauged Thirring model, see [5] for more details. It is well known that the BRST invariant total Lagrangian,

$$\mathcal{L}_{Th'''} = \mathcal{L}_{Th''} + \mathcal{L}_{GF} + \mathcal{L}_{FP}, \tag{2.6}$$

is obtained by adding the gauge-fixing term and the Faddeev-Popov (FP) ghost term $\mathcal{L}_{GF+FP}$ to the Lagrangian $\mathcal{L}_{Th''}$. The nilpotent BRST transformation ($\delta_B^2 * = 0$) is given by

$$\begin{aligned}
\delta_B \Psi^j(x) &= iC(x)\Psi^j(x), \\
\delta_B A_\mu(x) &= \partial_\mu C(x), \\
\delta_B \theta(x) &= C(x), \\
\delta_B B(x) &= 0, \\
\delta_B C(x) &= 0, \\
\delta_B \bar{C}(x) &= iB(x).
\end{aligned} \tag{2.7}$$

Then the BRST invariance of the additional term is guaranteed by the construction:

$$\begin{aligned}
\mathcal{L}_{GF+FP}[A, \theta, C, \bar{C}, B] &= -i\delta_B(\bar{C}(F[A,\theta] + \frac{\xi}{2}B)) \\
&= BF[A,\theta] + \frac{\xi}{2}B^2 + i\bar{C}\delta_B F[A,\theta],
\end{aligned} \tag{2.8}$$

where $F[A,\theta]$ is the gauge-fixing condition. After integrating out the B field, we obtain

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(F[A,\theta])^2, \tag{2.9}$$

$$\mathcal{L}_{FP} = i\bar{C}\delta_B F[A,\theta] = i\bar{C}\left(\frac{\delta F[A,\theta]}{\delta A_\mu}\partial_\mu C(x) + \frac{\delta F[A,\theta]}{\delta\theta}C\right). \tag{2.10}$$

The covariant gauge is given by

$$F[A,\theta] = \partial^\mu A_\mu, \tag{2.11}$$

leading to

$$\mathcal{L}_{FP} = i\bar{C}\partial^\mu\partial_\mu. \tag{2.12}$$

For our purposes, the R_ξ gauge is more convenient:

$$F[A,\theta] = \partial^\mu A_\mu + \xi G^{-1}\theta, \tag{2.13}$$

so that the crossing term $-G^{-1}A_\mu\partial_\mu\theta$ coming from $\frac{1}{2G_T}(A_\mu - \partial_\mu\theta)^2$ is canceled with that of the gauge-fixing term $\mathcal{L}_{GF}$ and the total Lagrangian $\mathcal{L}_{Th'''}$ is decomposed into

three parts:

$$
\begin{aligned}
\mathcal{L}_{Th'''} &= \mathcal{L}_{\Psi,A} + \mathcal{L}_{\theta} + \mathcal{L}_{FP}, \\
\mathcal{L}_{\Psi,A} &= \bar{\Psi}^j i\gamma^\mu(\partial_\mu - ieA_\mu)\Psi^j - \bar{\Psi}^j \hat{m}_0 \Psi^j \\
&\quad + \frac{e^2 G_T^{-1}}{2}(A_\mu)^2 - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \\
\mathcal{L}_\theta &= \frac{G_T^{-1}}{2}(\partial_\mu\theta)^2 - \frac{\xi}{2}(G_T^{-1}\theta)^2, \\
\mathcal{L}_{FP} &= i\bar{C}\left(\partial_\mu\partial^\mu + \xi G_T^{-1}\right)C,
\end{aligned}
\tag{2.14}
$$

where we have rescaled the field A_μ as $A_\mu \to eA_\mu$. Note that the scalar field θ is completely decoupled independently of ξ in the R_ξ gauge. This is an advantage of taking the R_ξ gauge in this paper.

We find the following limiting cases [5]. In the limit $e^2 \to \infty$, this model reduces to the gauge invariant-reformulation of the (massive) Thirring model [6] where θ is nothing but the well-known Stückelberg field (or the Batalin-Fradkin field). In the limit $G_T \to \infty$, QED with N_f flavors is recovered: $\mathcal{L}_{QED} = \bar{\Psi}^j i\gamma^\mu(\partial_\mu - ieA_\mu)\Psi^j - \bar{\Psi}^j \hat{m}_0 \Psi^j - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$. The weak gauge-coupling limit $e \to 0$ of the gauged Thirring model reduces to the non-linear σ model. $\mathcal{L}_{NL\sigma} = \kappa|\partial_\mu\phi|^2 \quad (|\phi| = 1)$.

3 WT identity and SD equation

The usual (longitudinal) WT identity for the vector current

$$
\mathcal{J}_\mu(x) := \bar{\Psi}(x)\gamma_\mu\Psi(x) \tag{3.1}
$$

is given by

$$
\begin{aligned}
\partial_\mu \langle \mathcal{J}^\mu(x); \Psi(y); \bar{\Psi}(z)\rangle_c &= \langle \frac{1}{e}\partial_\mu \Delta^{\mu\rho}(\partial) A_\rho(x); \Psi(y); \bar{\Psi}(z)\rangle_c \\
&= \langle \Psi(y)\bar{\Psi}(z)\rangle_c \delta^D(x-z) - \langle \Psi(y)\bar{\Psi}(z)\rangle_c \delta^D(x-y),
\end{aligned}
\tag{3.2}
$$

where $\Delta^{\mu\rho}$ is the inverse gauge-boson propagator obtained from (2.14) and $\langle ... \rangle_c$ denotes the connected correlation function. For the proper fermion-boson vertex function in momentum representation

$$
S(q)\Gamma^\mu(q,p)S(p) := \int d^D y \int d^D z e^{i(q\cdot y - p\cdot z)} \langle \mathcal{J}^\mu(0); \Psi(y); \bar{\Psi}(z)\rangle_c, \tag{3.3}
$$

the well-known form of the WT identity is obtained from (3.2):

$$
k_\mu \Gamma^\mu(q,p) = S^{-1}(q) - S^{-1}(p), \quad k_\mu := q_\mu - p_\mu, \tag{3.4}
$$

where $S(p)$ is the full fermion propagator in momentum representation.

In the previous paper [2], the identity for the rotation of the vector current, which we call the *transverse* WT identity, has been rederived based on the path integral

formalism. Such a type of WT identities was first derived by Takahashi [3]. We have found that, in $D = 1+1$ dimensions, the transverse WT identity has the remarkably simple form: [2]

$$\begin{aligned} & \partial_\mu \langle \mathcal{J}_\nu(x); \Psi(y)\bar{\Psi}(z)\rangle_c - \partial_\nu \langle \mathcal{J}_\mu(x); \Psi(y)\bar{\Psi}(z)\rangle_c \\ = \ & \langle \bar{\Psi}(x)\{\sigma_{\mu\nu}, \hat{m}_0\}\Psi(x); \Psi(y)\bar{\Psi}(z)\rangle_c \\ & -\langle \Psi(y)\bar{\Psi}(x)\rangle_c \sigma_{\mu\nu}\delta^D(x-z) - \sigma_{\mu\nu}\langle \Psi(x)\bar{\Psi}(z)\rangle_c \delta^D(x-y), \end{aligned} \tag{3.5}$$

where [3] $\sigma_{\mu\nu} := \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. In the chiral limit $M = 0$, especially, the transverse WT identity leads to the surprisingly simple identity for the rotation of the vector vertex in $D = 2$ dimensions:

$$k_\mu \Gamma_\nu(q,p) - k_\nu \Gamma_\mu(q,p) = S^{-1}(q)\sigma_{\mu\nu} + \sigma_{\mu\nu}S^{-1}(p), \tag{3.6}$$

which is similar in form to the longitudinal part (3.4).

The bare gauge-boson propagator $D^{(0)}_{\mu\nu}(k)$ is written as

$$D^{(0)-1}_{\mu\nu}(k) = (k^2 - e^2 G_T^{-1})g_{\mu\nu} - k_\mu k_\nu + \xi(k^2)^{-1}k_\mu k_\nu, \tag{3.7}$$

if we adopt the (nonlocal) R_ξ gauge with (momentum-dependent) gauge-fixing parameter $\xi(k^2)$ [6, 5]. In momentum representation, the SD equation for the full gauge-boson propagator $D_{\mu\nu}(k)$ is given by

$$\begin{aligned} D^{-1}_{\mu\nu}(k) &= D^{(0)-1}_{\mu\nu}(k) - \Pi_{\mu\nu}(k), \\ \Pi_{\mu\nu}(k) &:= e^2 \int \frac{d^D p}{(2\pi)^D} \mathrm{tr}[\gamma_\mu S(p)\Gamma_\nu(p, p-k)S(p-k)]. \end{aligned} \tag{3.8}$$

In the gauge theory, the vacuum polarization tensor should have the transverse form:

$$\Pi_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Pi(k), \tag{3.9}$$

as long as the gauge invariance is preserved. Hence we can write the full gauge-boson propagator in the following form:

$$\begin{aligned} D_{\mu\nu}(k) &= D_T(k^2)\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \frac{\xi(k^2)}{k^2 - \xi(k^2)e^2 G_T^{-1}}\frac{k_\mu k_\nu}{k^2}, \\ D_T(k^2) &:= \frac{1}{k^2 - e^2 G_T^{-1} - \Pi(k)}. \end{aligned} \tag{3.10}$$

This decomposition of the full gauge-boson propagator into the transverse $D^T_{\mu\nu}(k)$ and the longitudinal part $D^L_{\mu\nu}(k)$ is most convenient for our purposes, since this decomposition is preserved in the SD equation for the fermion propagator as can be

[2] In two dimensions, the existence of chiral anomaly does not change the transverse WT identity [2].

[3] In 1+1 dimensional space-time, we choose $\gamma^0 = \sigma_2, \gamma^1 = i\sigma_1, \gamma^5 := \gamma^0\gamma^1 = \sigma_3, (\epsilon_{01} = 1)$, which implies $\sigma_{\mu\nu} = i\epsilon_{\mu\nu}\gamma_5$ and $\gamma_\mu\gamma_5 = \epsilon_{\mu\nu}\gamma^\nu$.

seen shortly. Since there are N_f identical fermions, the vacuum polarization in 1+1 dimensions is given by

$$\Pi(k) = N_f \frac{e^2}{\pi}, \tag{3.11}$$

which is consistent with the chiral anomaly in 1+1 dimensions, see [2]. This implies that the mass μ_A for the gauge field A_μ is dynamically generated and given by $\mu_A = e/\sqrt{\pi}$.

On the other hand, the SD equation for the full fermion propagator $S(p)$ is given by

$$S_0^{-1}(p)S(p) = 1 + ie^2 \int \frac{d^D k}{(2\pi)^D} \gamma^\mu D_{\mu\nu}(k) S(p-k) \Gamma_\nu(p-k,p) S(p), \tag{3.12}$$

where S_0 the bare fermion propagator:

$$S_0(p) := \frac{1}{\hat{p} - m_0}, \quad (\hat{p} := \gamma^\mu p_\mu). \tag{3.13}$$

As shown in the previous paper [2], the transverse and the longitudinal WT identities can specify the vertex function which appears in the *integrand* of the SD equation for the fermion propagator. Note that

$$\begin{aligned} & D_{\mu\nu}(k)\tilde{\Gamma}^\nu(q,p) \\ = \;& D^L_{\mu\nu}(k)\tilde{\Gamma}^\nu(q,p) + D^T_{\mu\nu}(k)\tilde{\Gamma}^\nu(q,p) \\ = \;& \frac{\xi(k^2)}{k^2 - \xi(k^2)e^2 G_T^{-1}} \frac{k_\mu}{k^2}[k_\nu \tilde{\Gamma}^\nu(q,p)] + D_T(k^2)\frac{k^\nu}{k^2}[k_\nu \tilde{\Gamma}_\mu(q,p) - k_\mu \tilde{\Gamma}_\nu(q,p)], \end{aligned} \tag{3.14}$$

where we have defined $q := p - k$ and

$$\tilde{\Gamma}_\nu(q,p) := S(q)\Gamma_\nu(q,p)S(p). \tag{3.15}$$

Substituting the longitudinal WT identity (3.4) and the transverse WT identity (3.6) into (3.14), we get the exact SD equation [2] for the fermion propagator (in the chiral limit $m_0 = 0$):

$$\hat{p}S(p) = 1 + ie^2 \int \frac{d^2k}{(2\pi)^2} S(p-k)\hat{k}\left[\frac{D_T(k^2) - L_\xi(k^2)}{k^2}\right], \tag{3.16}$$

where $\hat{p} := \gamma^\mu p_\mu$, $\hat{k} := \gamma^\mu k_\mu$ and

$$D_T(k^2) := \frac{1}{k^2 - e^2 G_T^{-1} - e^2 N_f/\pi}, \quad L_\xi(k^2) := \frac{\xi(k^2)}{k^2 - \xi(k^2)e^2 G_T^{-1}}. \tag{3.17}$$

This SD equation is exact in 1+1 space-time dimensions. The solution is easily found by moving to the coordinate space, since the Fourier transformation changes the convolution in momentum space into the simple product in the coordinate space:

$$i\hat{\partial}S(x) = \delta^2(x) - ie^2 S(x) \int \frac{d^2k}{(2\pi)^2} \hat{k}\left[\frac{D_T(k^2) - L_\xi(k^2)}{k^2}\right] e^{-ik\cdot x}. \tag{3.18}$$

This equation can be solved exactly. Thus the full fermion propagator of the 1+1 dimensional (Abelian-)gauged Thirring model in the R_ξ gauge (2.13) is given by

$$S(x) = S_0(x) \exp \left\{ -ie^2 \int \frac{d^2k}{(2\pi)^2} \left[\frac{D_T(k^2) - L_\xi(k^2)}{k^2} \right] (e^{-ik\cdot x} - 1) \right\}, \tag{3.19}$$

where we have assumed the translational invariance $S(x, y) = S(x-y, 0) := S(x-y)$ and the massless bare fermion propagator S_0 is given by

$$S_0(p) := \frac{1}{\hat{p}}, \quad S_0(x) = \int \frac{d^2p}{(2\pi)^2} e^{ip\cdot x} S_0(p) = \frac{1}{2\pi} \frac{\hat{x}}{x^2}. \tag{3.20}$$

In the limit $G_T \to \infty$, (3.19) reproduces the exact solution of massless QED$_2$ (Schwinger model) [11] (see also the Appendix of [12]):

$$S(x) = S_0(x) \exp \left\{ -ie^2 \int \frac{d^2k}{(2\pi)^2} \left[\frac{1}{k^2(k^2 - N_f e^2/\pi)} - \frac{\xi(k^2)}{k^4} \right] (e^{-ik\cdot x} - 1) \right\}, \tag{3.21}$$

which agrees also with Stam's result [13] based on the gauge technique.

4 Thirring model as strong coupling limit

The pure Thirring model is recovered in the strong gauge-coupling limit $e^2 \to \infty$. In this limit, the gauge-independent part D_T reads

$$e^2 D_T(k^2) \to -\frac{1}{G_T^{-1} + N_f/\pi} = -\frac{G_T}{1 + N_f G_T/\pi} := -\lambda_0. \tag{4.1}$$

On the other hand, for the gauge-dependent part L_ξ, we consider the limit $\xi \to \pm\infty$ which corresponds to the unitary gauge, $\theta = 0$. This gauge should reproduce the original Thirring model (at least in the quenched limit $N_f \to 0$ or in the absence of quantum correction of the gauge-boson propagator). In the limit $\xi \to \pm\infty$, we see $L_\xi(k^2) \to -G_T$, and hence

$$\frac{e^2 D_T(k^2) - L_\xi(k^2)}{k^2} \to -\frac{\lambda_\infty}{k^2}, \tag{4.2}$$

where

$$\lambda_\infty := \lambda_0 - G_T = -\frac{N_f G_T^2/\pi}{1 + N_f G_T/\pi}. \tag{4.3}$$

Thus we get the exact solution in the gauge $\xi \to \pm\infty$ as

$$S(x) = S_0(x) \exp \left\{ i\lambda_\infty \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2} (e^{-ik\cdot x} - 1) \right\}, \tag{4.4}$$

which agrees with Johnson's result [14]. The solution in the $\xi = 0$ gauge is obtained by replacing λ_∞ with λ_0 in (4.4).

5 SD equation in the momentum space

As proposed in the previous paper [2], rather than the inverse propagator

$$S(p)^{-1} = A(p)\hat{p} - B(p), \tag{5.1}$$

it is more convenient to use the following decomposition in the SD equation for $S(p)$:

$$S(p) = \frac{\hat{p}X(p) + Y(p)}{p^2}, \tag{5.2}$$

and solve a pair of integral equations for X and Y. In general, X and Y couple each other. Nevertheless, the equations are still linear in each variable, X or Y, and hence we do not need to take the linearization approximation, which is in sharp contrast with the pair of equations for A and B obtained from the decomposition of the SD equation for $S^{-1}(p)$, see [2]. The solution, X and Y for such a pair of equations may include the non-linear effect, since X and Y are related with A and B as follows:

$$X(p) := \frac{A(p)p^2}{A^2(p)p^2 + B^2(p)}, \quad Y(p) := \frac{B(p)p^2}{A^2(p)p^2 + B^2(p)}. \tag{5.3}$$

The wavefunction renormalization $A(p)$ and the mass function $M(p)$ is obtained from X and Y as

$$M(p) := \frac{B(p)}{A(p)} = \frac{Y(p)}{X(p)}, \quad Z(p) := \frac{1}{A(p)} = X(p)\left(1 + \frac{M^2(p)}{p^2}\right). \tag{5.4}$$

The chiral order parameter is determined simply as

$$\langle\bar{\Psi}\Psi\rangle/N_f = \int \frac{d^2p}{(2\pi)^2}\mathrm{tr}[S(p)] = \mathrm{tr}(1)\int_0^{\Lambda^2} \frac{dp^2}{4\pi}\frac{Y(p)}{p^2}. \tag{5.5}$$

In 1+1 dimensions, especially, the SD equation (3.16) is linear in S and can be reduced to a decoupled pair of integral equations for X and Y:

$$\begin{aligned} X(p) &= 1 + \int \frac{d^2k}{(2\pi)^2}\frac{k\cdot(p-k)}{(p-k)^2}\left[\frac{e^2D_T(k^2) - L_\xi(k^2)}{k^2}\right]X(p-k), \\ Y(p) &= \int \frac{d^2k}{(2\pi)^2}\frac{p\cdot k}{(p-k)^2}\left[\frac{e^2D_T(k^2) - L_\xi(k^2)}{k^2}\right]Y(p-k), \end{aligned} \tag{5.6}$$

which can be solved independently.

6 Dynamical mass generation?

To look for the solution representing the dynamical mass generation, we move back to the Euclidean space and solve the SD equation (5.6) in momentum space. The

solution of (5.6) is easily obtained especially in the gauge $\xi = 0, \pm\infty$. After change of variable $p - k = q$, Eq. (5.6) in the pure Thirring model limit $e^2 \to \infty$ reads

$$\begin{aligned} X(p) &= 1 + \int \frac{d^2q}{(2\pi)^2} \frac{(p-q)\cdot q}{q^2} \left[\frac{\lambda}{(p-q)^2} \right] X(q), \\ Y(p) &= \int \frac{d^2q}{(2\pi)^2} \frac{p\cdot(p-q)}{q^2} \left[\frac{\lambda}{(p-q)^2} \right] Y(q), \end{aligned} \tag{6.1}$$

where λ denotes λ_0 or λ_∞ corresponding to the gauge $\xi = 0$ or $\pm\infty$, respectively. After angular integration, (6.1) reduces to

$$X(p) = 1 - \frac{\lambda}{4\pi} \int_{p^2}^{\infty} \frac{dq^2}{q^2} X(q), \quad Y(p) = \frac{\lambda}{4\pi} \int_0^{p^2} \frac{dq^2}{q^2} Y(q). \tag{6.2}$$

These are integral equation of the Volterra type and can be converted to the first order differential equation

$$\frac{d}{dp^2} X(p) = \frac{\lambda}{4\pi} \frac{X(p)}{p^2}, \quad \frac{d}{dp^2} Y(p) = \frac{\lambda}{4\pi} \frac{Y(p)}{p^2}, \tag{6.3}$$

with a boundary (or initial) condition, $X(\infty) = 1, Y(0) = 0$. The solution of the differential equation is obtained as

$$X(p) = C_1 (p^2)^{\frac{\lambda}{4\pi}}, \quad Y(p) = C_2 (p^2)^{\frac{\lambda}{4\pi}}, \tag{6.4}$$

with constants C_1, C_2, which imply

$$M(p) := \frac{B(p)}{A(p)} = \frac{C_2}{C_1} =: M, \quad Z(p) := \frac{1}{A(p)} = X(p) \left(1 + \frac{M}{p^2} \right). \tag{6.5}$$

The infrared (IR) boundary condition for $Y(p)$: $Y(p = 0) = 0$ is satisfied by an arbitrary constant $C_2 = MC_1$. Note that the dynamical fermion mass function $M(p)$ is p-independent constant M. If there is no cutoff, $X(p)$ has no finite solution. By introducing the ultraviolet (UV) cutoff Λ, the UV boundary condition $X(\Lambda) = 1$ determines the solution uniquely $C_1 = (\Lambda^2)^{\frac{-\lambda}{4\pi}}$, i.e.

$$X(p) = \left(\frac{p^2}{\Lambda^2} \right)^{\frac{\lambda}{4\pi}}, \quad Y(p) = MX(p), \quad S(p) = \frac{\hat{p} + M}{p^2} X(p). \tag{6.6}$$

When there is no dynamical mass generation for the fermion $Y(p) = 0$ (i.e. $M = 0$), the wavefunction renormalization function $A(p)$ has a solution given by

$$A_0(p) = X^{-1}(p) = \left(\frac{p^2}{\Lambda^2} \right)^{-\frac{\lambda}{4\pi}} = \exp\left[-\frac{\lambda}{4\pi} \ln \frac{p^2}{\Lambda^2} \right]. \tag{6.7}$$

Therefore, the SD equation (6.1) has a consistent solution (6.6) which corresponds to no dynamical mass generation $M = 0$.

Next, we want to ask whether the SD equation has a solution of the dynamical mass generation, i.e. $M \neq 0$ even for $m_0 = 0$. In two dimensions, there is no spontaneous breakdown of *continuous* symmetry due to Mermin-Wagner [15] and Coleman's theorem [16]. Therefore, the U(1) chiral symmetry $\Psi \to e^{i\gamma_5\alpha}\Psi$ is not broken spontaneously and the chiral order parameter vanishes identically in two dimensions. This does not necessarily imply that the fermion remains massless. Actually, it is possible for the fermion to acquire a mass according to Witten [17]. From the solution obtained above, we find that the following identity (corresponding to Eq. (5.5)) holds

$$M = \lambda \int_0^{\Lambda^2} \frac{dp^2}{4\pi} \frac{M}{p^2} X(p), \quad X(p) = \exp\left[\frac{\lambda}{4\pi} \ln \frac{p^2}{\Lambda^2}\right]. \tag{6.8}$$

If we truncate this equation up to some finite order of $1/N_f$ expansion where $N_f G$ is fixed to be a constant (i.e. $\lambda = O(1/N_f)$), we can get the non-trivial scaling law. Actually, in the leading order of $1/N_f$, we get the scaling of the essential singularity type (at the origin $\lambda = 0$):

$$\frac{\mu}{\Lambda} = e^{-\frac{2\pi}{\lambda}}, \tag{6.9}$$

where we have introduced the infrared cutoff μ in Eq. (6.8) as a lower bound of integration which plays the role of the dynamically generated mass. This scaling shows the asymptotic freedom in two dimensions [18].

For finite e^2, finally, we find

$$\begin{aligned} & ie^2 \int \frac{d^2k}{(2\pi)^2} \left[\frac{D_T(k^2) - L_\xi(k^2)}{k^2}\right] e^{-ik\cdot x} \\ = \quad & -i\lambda_0[\Delta_F(x;0) - \Delta_F(x; e^2\lambda_0^{-1})] + iG[\Delta_F(x;0) - \Delta_F(x;\xi G^{-1})]. \end{aligned} \tag{6.10}$$

This shows that the difference of this equation with that in the limit $e = \infty$ is just $i\lambda_0 \Delta_F(x; e^2\lambda_0^{-1})$. Hence Eq. (5.6) reads

$$\begin{aligned} X(p) &= 1 + \int \frac{d^2q}{(2\pi)^2} \frac{(p-q)\cdot q}{q^2} \left[\frac{\lambda}{(p-q)^2} - \frac{\lambda_0}{(p-q)^2 + e^2\lambda_0^{-1}}\right] X(q), \\ Y(p) &= \int \frac{d^2q}{(2\pi)^2} \frac{p\cdot(p-q)}{q^2} \left[\frac{\lambda}{(p-q)^2} - \frac{\lambda_0}{(p-q)^2 + e^2\lambda_0^{-1}}\right] Y(q). \end{aligned} \tag{6.11}$$

However, it is rather difficult to find the exact solution for this equation. In a subsequent paper [19], we will give detailed study on dynamical mass generation in the case of finite e^2. Alternative approach to the gauged Thirring model is the bosonization technique by which the the apparent inconsistency (dynamical fermion mass generation and asymptotic freedom without breaking the continuous chiral symmetry) is clearly understood, which will be given elsewhere [20]. Further studies of the gauged Thirring model will enable us to compare the results with the lattice version of this model investigated recently by Jersak et al. [21].

Acknowledgments

The author would like to thank Prof. Ian J.R. Aitchison for kind hospitality in Oxford. This work is supported in part by the Japan Society for the Promotion of Science and the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (No.07640377).

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